Dirac Particle in a Constant Magnetic Field: Path Integral Treatment

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The Green functions related to a Dirac particle in a constant magnetic field are calculated via two methods, global and local, by using the supersymmetric formalism of Fradkin and Gitman. The energy spectrum as well as the corresponding wave functions are extracted following these two approaches.

Key words: Path Integral; Dirac Equation; Propagator.

1. Introduction

The path integral was of great benefit when in trying to bring back the quantum description of a physical system to its classical analogue. However, it has been shown to be inconvenient for the case of the description of the spin, which is a purely discrete physical entity without a classical analogue, unlike the path integral, which is principally based on classical images such as trajectories. In attempting to solve this difficulty, there have been essentially two categories of path integral formulation. The first one, called bosonic model, proposes to use commuting variables to describe the spin dynamics and the second one, known as fermionic model, uses the Grassmann (anticommuting) variables. This latter model was proposed by Fradkin and Gitman [1] who have presented the Dirac propagator by means of a Grassmannian path integral and have established a rigorous formulation of this path integral, which has various properties such as gauge invariance, reparametrization, invariance and supersymmetric form [1, 2].

This attempt was of remarkable impact on the research of analytical and exact expressions of the relativistic spinning propagators in the presence of external fields as such many problems have been resolved, namely the interaction with a plane wave [3 – 5] as well as the case of the interaction with a constant electromagnetic field [6, 7]. However, in the latter case it is not possible to extract solutions for the constant magnetic field alone ($\vec{E} = \vec{0}$), and for that reason we are required to repeat the same set of calculations of the Dirac propagator, which would bring us nowhere, due to the fact that a matrix $F_{\mu\nu} (= \partial_\mu A_\nu - \partial_\nu A_\mu)$ becomes non-defined where $\vec{E} = \vec{0}$.

Many attempts have been made to overcome this problem, which were most often restricted to the case of the second-order Dirac equation [8]. However, a solution of the first-order Dirac equation for a constant magnetic field is still non-existent.

Thus, in this paper we propose to give the solution for a relativistic Dirac particle in a constant magnetic field in two different manners. The first one, the so-called local approach, is related to the first-order Dirac equation formulation, where there is no operator and the states are all physical without any being superfluous. The second one, the so-called global approach, is related to the square of the Dirac equation, where the superfluous or non-physical states are eliminated thereafter by activating an operator [9]. The treatment we aim at is analytical and exact. The Green function is expressed in terms of a nucleus of a standard form and of a Feymann type in $\exp(\mathrm{i}S)$, where S is a supersymmetric action.

Our aim in treating this elementary problem of a relativistic half spin particle submitted to a constant magnetic field directed towards the z-axis is to show how to extract the energy spectrum and its corresponding wave functions in terms of the Green function.

The calculations we are undertaking lead at a given stage to the calculus of a harmonic oscillator with more difficulty when compared with [10].

The propagator of the Dirac particle in an external electromagnetic field is the causal Green function

$S^{\mathrm{c}}(x_b, x_a)$ of the Dirac equation

$$(\gamma.\pi_b - m)S^{\mathrm{c}}(x_b, x_a) = -\delta^4(x_b - x_a), \qquad (1)$$

where $\pi_\mu = (\mathrm{i}\partial_\mu - gA_\mu)$; g is the electronic charge; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$; $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_-$; $[\gamma^\mu, \gamma^\nu]+ = 2\eta^{\mu\nu}$; $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$; $\mu, \nu = \overline{0, 3}$.

The scalar product, denoted by a dot, stands for $a.b = a_{.\mu} b^\mu$.

Multiplying by γ^5 on both sides of (1), we get

$$(\widetilde{\gamma}.\pi_b - m\gamma^5)\widetilde{S}(x_b, x_a) = \delta^4(x_b - x_a), \qquad (2)$$

where $\widetilde{S} = S^{\mathrm{c}}\gamma^5$; $\widetilde{\gamma}^\mu = \gamma^5\gamma^\mu$; $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \widetilde{\gamma}^5$; $(\gamma^5)^2 = -1$. The matrices $\widetilde{\gamma}^\mu$ have the same commutation relations as the initial ones γ^μ; $[\gamma^n, \gamma^m]_+ = 2\eta^{nm}$; $n, m = \overline{0, 3}, 5$; $\eta^{nm} = \mathrm{diag}(1, -1, -1, -1, -1)$.

Formally, $\widetilde{S}(x_b, x_a)$ is the matrix element in the coordinate space of the inverse Dirac operator, namely

$$\widetilde{S}(x_b, x_a) = \langle x_b | \widetilde{S} | x_a \rangle, \qquad (3)$$

with

$$\widetilde{S} = (\widetilde{\gamma}^\mu \pi_\mu - m\gamma^5)^{-1} = O^{-1}. \qquad (4)$$

Using the Schwinger trick, we represent $\widetilde{S}$ by the integral expressions

$$\begin{aligned} \widetilde{S} = \widetilde{S}_{\mathrm{l}} &= S_{\mathrm{l}}\gamma^5 \\ &= \int_0^\infty \mathrm{d}e \int \exp\left[\mathrm{i}e\left(O^2 + \mathrm{i}\varepsilon\right) + \chi O\right] \mathrm{d}\chi, \\ \widetilde{S} = \widetilde{S}_{\mathrm{g}} &= S_{\mathrm{g}}\gamma^5 = O G_{\mathrm{g}} \\ &= O\left(-\mathrm{i}\int_0^\infty \mathrm{d}e \exp\left[\mathrm{i}e\left(O^2 + \mathrm{i}\varepsilon\right)\right]\right), \end{aligned} \qquad (5)$$

where $\widetilde{S}_{\mathrm{l}}$ is called the local representation and $\widetilde{S}_{\mathrm{g}}$ the global one. It is clear that G_{g} verifies the following quadratic Dirac equation:

$$O_b^2 G_{\mathrm{g}}(x_b, x_a) = \delta^4(x_b - x_a), \qquad (6)$$

with

$$O^2 = \pi_b^2 - m^2 - \frac{\mathrm{i}g}{2} F_{\alpha\beta} \widetilde{\gamma}^\alpha \widetilde{\gamma}^\beta,$$

$\widetilde{S}_{\mathrm{g}}(x_b, x_a)$ can be obtained starting from $G_{\mathrm{g}}(x_b, x_a)$:

$$\widetilde{S}_{\mathrm{g}}(x_b, x_a) = (\widetilde{\gamma} \cdot \pi_b - m\gamma^5) G_{\mathrm{g}}(x_b, x_a). \qquad (7)$$

Doing so, we formulate the problem in the two projections: local and global. In the case of the local projection the operator projection O is replaced by a path integral using a fermionic proper time χ [1], while for the global one this operator shall act at the end of the evolution to eliminate the superfluous states corresponding to the square of the Dirac operator, and this latter is written in the path integral representation following [9].

It is remarkable to see that the calculations of the global description G_{g} are similar to those of the local one $\widetilde{S}_{\mathrm{l}}$ before integrating over the proper time χ. Consequently, it is convenient to unify these two Green functions into a λ-modified one $\widetilde{S}_\lambda$ [3–5] by taking into account the λ-modified measure of χ, which ensures that

$$\widetilde{S}_\lambda = \begin{cases} \widetilde{S}_{\mathrm{l}}, & \lambda = 1, \\ G_{\mathrm{g}}, & \lambda = 0. \end{cases} \qquad (8)$$

In fact, knowing that $\int \mathrm{d}\chi = 0$ and $\int \mathrm{d}\chi\chi = 1$, we easily write for the Green function $\widetilde{S}_\lambda$ the following result [1, 9]:

$$\begin{aligned} \widetilde{S}_\lambda = \left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda} \exp\left(\mathrm{i}\widetilde{\gamma}.\frac{\partial_{\mathrm{l}}}{\partial\theta}\right) \int_0^\infty \mathrm{d}e \int \mathrm{d}\chi\chi^{1-\lambda} \int Dx\mathcal{D}\Psi\mathbb{M}(e) \exp\Bigg\{\mathrm{i}\int_0^1 \Bigg[\frac{-\dot{x}^2}{2e} - \frac{e}{2}m^2 - g\dot{x}.A \\ + \mathrm{i}eg\Psi.F.\Psi + \mathrm{i}\lambda\left(\frac{\dot{x}.\Psi}{e} - m\Psi^5\right)\chi - \mathrm{i}\Psi.\dot{\Psi}\Bigg] \mathrm{d}\tau + \Psi(1).\Psi(0)\Bigg\}\Bigg|_{\theta=0}, \end{aligned} \qquad (9)$$

where x, p, e and χ, Ψ^n, θ^n are even and odd variables, respectively, and satisfy the following boundary conditions:

$$x(0) = x_a, \quad x(1) = x_b, \qquad \Psi^n(0) + \Psi^n(1) = \theta^n,$$

whereas the measures $\mathbb{M}(e)$ and $\mathcal{D}\Psi$ are defined as

$$\mathbb{M}(e) = \int D\pi \exp\left(\frac{\mathrm{i}}{2}\int_0^1 e\pi^2 \mathrm{d}\tau\right), \qquad \mathcal{D}\Psi = D\Psi \left[\int_{\Psi(1)+\Psi(0)=0} D\Psi \exp\left(\int_0^1 \Psi_n \dot{\Psi}^n \mathrm{d}\tau\right)\right]^{-1}.$$

The aim of this paper is to calculate this Green function $\widetilde{S}_\lambda$ in the case of a constant magnetic field.

For convenience we choose the interaction by the quadri-potential

$$A_0 = A_2 = A_3 = 0, \quad A_1 = Bx^2 = By. \tag{10}$$

2. Energy Spectra and Wave Functions

The Green function given in (9) is in its Lagrangian form. By linearizing the Gaussian related to the kinetic term, we get the Green function in its Hamiltonian form, which is more convenient to use in our calculations:

$$\begin{aligned}\widetilde{S}_\lambda = &\left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda} \exp\left(\mathrm{i}\widetilde{\gamma}.\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e \int \mathrm{d}\chi\chi^{1-\lambda}\int DxDp\mathcal{D}\Psi \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau \left[p.\dot{x} + \frac{e}{2}\left(p^2 - m^2\right)\right.\right.\\ &\left.\left.\left. + egp.A + \frac{e}{2}g^2A^2 + \mathrm{i}eg\Psi.F.\Psi - \mathrm{i}\lambda[(p+gA).\Psi + m\Psi^5]\chi - \mathrm{i}\Psi_n\dot{\Psi}^n\right] + \Psi_n(1)\Psi^n(0)\right\}\right|_{\theta=0}.\end{aligned} \tag{11}$$

By introducing the characteristics of the field, $\widetilde{S}_\lambda$ becomes

$$\begin{aligned}\widetilde{S}_\lambda = &\left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda} \exp\left(\mathrm{i}\widetilde{\gamma}.\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e \int \mathrm{d}\chi\chi^{1-\lambda}\int \mathcal{D}\Psi \int Dp_0Dp_1Dp_2Dp_3DtDxDyDz\\ &\cdot \exp\left\{\mathrm{i}\int_0^1 \left[p_0\dot{t} + p_1\dot{x} + p_3\dot{z} + p_2\dot{y} + \frac{e}{2}\left(p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2\right) - \frac{e}{2}g^2B^2y^2 - egBp_1y\right.\right.\\ &\left.\left.\left.-\mathrm{i}\lambda gBy\tilde{W}_1.\widetilde{\Psi}\chi - \mathrm{i}\lambda\left(p.\Psi + m\Psi^5\right)\chi + egB\widetilde{\Psi}.\sigma_2.\widetilde{\Psi} - \mathrm{i}\Psi_n\dot{\Psi}^n\right]\mathrm{d}\tau + \Psi_n(1)\Psi^n(0)\right\}\right|_{\theta=0},\end{aligned} \tag{12}$$

where $\widetilde{\Psi} = \begin{pmatrix}\Psi^1\\ \Psi^2\end{pmatrix}$, $\tilde{W}_1 = (1,0)$ and $\sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}$.

By integrating over the paths t, x and z, we can see that the momentums become motion constants:

$$p_0 = \mathrm{const}, \; p_1 = \mathrm{const}, \; p_3 = \mathrm{const}. \tag{13}$$

By integrating then over p_2, we rewrite $\widetilde{S}_\lambda$ as

$$\begin{aligned}\widetilde{S}_\lambda = &\left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda} \exp\left(\mathrm{i}\widetilde{\gamma}.\frac{\partial_l}{\partial\theta}\right)\int_0^\infty \mathrm{d}e \int \mathrm{d}\chi\chi^{1-\lambda}\int \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\int \mathcal{D}\Psi \exp\left\{\mathrm{i}p_0(t_b - t_a) + \mathrm{i}p_1(x_b - x_a)\right.\\ &+ \mathrm{i}p_3(z_b - z_a) + \frac{\mathrm{i}e}{2}(p_0^2 - p_3^2 - m^2) + \frac{\lambda}{e}\tilde{W}_2.(y_b\widetilde{\Psi}_b - y_a\widetilde{\Psi}_a)\chi + \frac{\lambda p_1}{egB}\tilde{W}_2.(\widetilde{\Psi}_b - \widetilde{\Psi}_a)\chi\\ &\left.+ \mathrm{i}\int_0^1 \mathrm{d}\tau\left[egB\widetilde{\Psi}.\sigma_2.\widetilde{\Psi} - \mathrm{i}\Psi_n\dot{\Psi}^n - \mathrm{i}\lambda(p_0\Psi^0 + p_3\Psi^3 + m\Psi^5)\chi\right] + \Psi_n(1)\Psi^n(0)\right\}\\ &\cdot\left.\int Dy \exp\left\{\mathrm{i}\int_0^1 \mathrm{d}\tau\left[\frac{M}{2}\dot{y}^2 - \frac{M\Omega^2}{2}\left(y + \frac{p_1}{gB}\right)^2 + f(\tau)\left(y + \frac{p_1}{gB}\right)\right]\right\}\right|_{\theta=0},\end{aligned} \tag{14}$$

where we have used the notations $M = \frac{1}{e}$, $\Omega = e|g|B$, $f(\tau) = -\mathrm{i}\lambda(gB\tilde{W}_1.\widetilde{\Psi} - \frac{1}{e}\tilde{W}_2.\dot{\widetilde{\Psi}})\chi$ and $\tilde{W}_2 = (0,1)$.

It is clear that the nucleus related to the $y(\tau)$ paths expresses the motion of a particle submitted to a harmonic strength. By using its known expression, namely its spectral decomposition [11], we obtain the following series:

$$\begin{aligned}\widetilde{S}_\lambda = &\left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda} \exp\left(\mathrm{i}\widetilde{\gamma}.\frac{\partial_1}{\partial\theta}\right)\int_0^\infty \mathrm{d}e\int \mathrm{d}\chi\chi^{1-\lambda}\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\int\mathcal{D}\Psi\sum_{n=0}^{+\infty}\left(\frac{|g|B}{\pi}\right)^{1/2}\frac{1}{2^n n!}\,\mathrm{e}^{-\mathrm{i}e|g|B\left(n+\frac{1}{2}\right)}\\
&\cdot \mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)} H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right)H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)\\
&\cdot\exp\left\{\mathrm{i}p_0\left(t_b-t_a\right)+\mathrm{i}p_1\left(x_b-x_a\right)+\mathrm{i}p_3\left(z_b-z_a\right)+\frac{\mathrm{i}e}{2}\left(p_0^2-p_3^2-m^2\right)+\frac{\lambda}{e}\tilde{W}_2.\left(y_b\widetilde{\Psi}_b-y_a\widetilde{\Psi}_a\right)\chi\right.\\
&+\frac{\lambda p_1}{egB}\tilde{W}_2.\left(\widetilde{\Psi}_b-\widetilde{\Psi}_a\right)\chi+\Psi_n(1)\Psi^n(0)+\mathrm{i}\int_0^1\left[egB\widetilde{\Psi}.\sigma_2.\widetilde{\Psi}-\mathrm{i}\Psi_n\dot{\Psi}^n-\mathrm{i}\lambda\left(p_0\Psi^0+p_3\Psi^3+m\Psi^5\right)\chi\right.\\
&\left.\left.+\frac{\left(y_b+\frac{p_1}{gB}\right)}{\sin\left(e|g|B\right)}f(\tau)\sin(e|g|B\tau)+\frac{\left(y_a+\frac{p_1}{gB}\right)}{\sin(e|g|B)}f(\tau)\sin(e|g|B(1-\tau))\right]\mathrm{d}\tau\right\}\Bigg|_{\theta=0}.\end{aligned}\tag{15}$$

We are left with the integration over the Grassmann variables.

To be able to undertake easily the integration over $\Psi(\tau)$, let us get rid of the antiperiodic condition $\Psi_n(1)+\Psi_n(0)=\theta$ by going through ω velocities via the change

$$\Psi^n(\tau)=\frac{1}{2}\int_0^1\varepsilon(\tau-\tau')\omega^n(\tau')\mathrm{d}\tau'+\frac{1}{2}\theta^n,\ \varepsilon(\tau)=\text{sign of }\tau.\tag{16}$$

With this change, the Green function becomes

$$\begin{aligned}\widetilde{S}_\lambda = &\left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda}\exp(\mathrm{i}\widetilde{\gamma}.\frac{\partial_1}{\partial\theta})\int_0^\infty\mathrm{d}e\int\mathrm{d}\chi\chi^{1-\lambda}\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\sum_{n=0}^{+\infty}\left(\frac{|g|B}{\pi}\right)^{1/2}\frac{1}{2^n n!}\\
&\cdot\mathrm{e}^{-\mathrm{i}e|g|B\left(n+\frac{1}{2}\right)}\mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)}H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right)H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)\\
&\cdot\exp\left\{\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_3(z_b-z_a)+\frac{ie}{2}(p_0^2-p_3^2-m^2)+\frac{\mathrm{i}egB}{4}\theta\sigma_2\theta\right.\\
&+\frac{\lambda}{2e}(y_b-y_a)\tilde{W}_2.\widetilde{\theta}\chi+\frac{\lambda}{2}(p_0\theta^0+p_3\theta^3+m\theta^5)\chi+\frac{\lambda gB\tilde{W}_1.\tilde{\theta}\chi}{2\sin\left(egB\right)}\int_0^1\left[\left(y_b+\frac{P_1}{gB}\right)\sin(egB\tau)\right.\\
&\left.\left.+\left(y_a+\frac{P_1}{gB}\right)\sin(egB(1-\tau))\right]\mathrm{d}\tau\right\}\mathcal{I}(\theta)\Bigg|_{\theta=0},\end{aligned}\tag{17}$$

where we have

$$\begin{aligned}\mathcal{I}(\theta)=&\int\mathcal{D}\omega^0\mathcal{D}\omega^3\mathcal{D}\omega^5\mathcal{D}\tilde{\omega}\exp\left[\frac{1}{2}\int_0^1\int_0^1\left(-\omega^0\varepsilon\omega^0+\omega^3\varepsilon\omega^3+\omega^5\varepsilon\omega^5+\tilde{\omega}.\mathcal{R}.\tilde{\omega}\right)\mathrm{d}\tau\mathrm{d}\tau'\right]\\
&\cdot\exp\left[\int_0^1\left(\mathcal{J}_0\omega^0+\mathcal{J}_3\omega^3+\mathcal{J}_5\omega^5+\widetilde{\mathcal{J}}\tilde{\omega}\right)\mathrm{d}\tau\right],\end{aligned}\tag{18}$$

$$\mathcal{D}\omega=\left[\int D\omega\exp\left(\int_0^1\omega_n\varepsilon\omega^n\mathrm{d}\tau\right)\right]^{-1}D\omega,\quad\tilde{\omega}=\begin{pmatrix}\omega^1\\\omega^2\end{pmatrix}\text{ and }\theta=\begin{pmatrix}\theta^1\\\theta^2\end{pmatrix}.$$

The currents associated to the velocities are

$$\begin{aligned}&\mathcal{J}_0(\tau)=-\frac{\lambda p_0}{2}\int_0^1\varepsilon(\tau'-\tau)\mathrm{d}\tau'\chi,\quad\mathcal{J}_3(\tau)=-\frac{\lambda p_3}{2}\int_0^1\varepsilon(\tau'-\tau)\mathrm{d}\tau'\chi,\\
&\mathcal{J}_5(\tau)=-\mathrm{i}\lambda m\int_0^1\varepsilon(\tau'-\tau)\mathrm{d}\tau'\chi,\quad\widetilde{\mathcal{J}}(\tau)=\frac{\mathrm{i}egB}{2}\theta\sigma_2\int_0^1\varepsilon(\tau'-\tau)\mathrm{d}\tau'+\lambda J(\tau)\chi,\end{aligned}\tag{19}$$

where

$$J(\tau)=-\frac{p_1}{egB}\tilde{W}_2-\frac{1}{2e}y_b\tilde{W}_2-\frac{1}{2e}y_a\tilde{W}_2+\frac{\mathrm{i}\left(y_b+\frac{p_1}{gB}\right)}{\sin(egB)}\left[-\frac{\mathrm{i}}{e}\tilde{W}_2\sin(egB\tau)\right.$$
$$\left.+\frac{\mathrm{i}gB}{2}\tilde{W}_1\left(\frac{1-\cos(egB)}{egB}-2\frac{1-\cos(egB\tau)}{egB}\right)\right]+\frac{\mathrm{i}\left(y_a+\frac{p_1}{gB}\right)}{\sin(egB)}\left[-\frac{\mathrm{i}}{e}\tilde{W}_2\sin(egB(1-\tau))\right.$$
$$\left.+\frac{\mathrm{i}gB}{2}\tilde{W}_1\left(\frac{1-\cos(egB)}{egB}-2\frac{\cos(egB(1-\tau))-\cos(egB)}{egB}\right)\right], \tag{20}$$

and the matrices are defined by

$$\mathcal{R}(\tau,\tau')=\int_0^1\varepsilon(\tau-\tau_1)\mathcal{G}(\tau_1,\tau')\mathrm{d}\tau_1,\quad \mathcal{G}(\tau_1,\tau')=\delta(\tau_1-\tau')I-\frac{\mathrm{i}eBg}{2}\varepsilon(\tau_1-\tau')\sigma_2. \tag{21}$$

The integral over the variable $\omega(\tau)$ is Gaussian and as the integrations over ω^0, ω^3, ω^5 give simple values 1. The result is done straightforwardly. It is equal to

$$\sqrt{\frac{\det\mathcal{R}}{\det\varepsilon}}\exp\left\{\frac{1}{2}\int_0^1\int_0^1\widetilde{\mathcal{J}}^n(\tau)\mathcal{R}^{-1}_{nm}(\tau,\tau')\widetilde{\mathcal{J}}^m(\tau')\mathrm{d}\tau\mathrm{d}\tau'\right\}. \tag{22}$$

We have

$$\det\mathcal{R}=\exp[\mathrm{tr}\ln\mathcal{R}]. \tag{23}$$

We can verify that

$$\sqrt{\frac{\det\mathcal{R}}{\det\varepsilon}}=\cos\left(\frac{egB}{2}\right). \tag{24}$$

We can see that the inverse matrix of $\mathcal{G}(\tau_1,\tau')$ [6] is

$$\mathcal{G}^{-1}(\tau_1,\tau)=\delta(\tau_1-\tau)+\frac{\mathrm{i}}{2}egB\sigma_2\mathrm{e}^{\mathrm{i}egB\sigma_2(\tau_1-\tau)}\left[\varepsilon(\tau_1-\tau)-\tanh\left(\frac{\mathrm{i}egB\sigma_2}{2}\right)\right], \tag{25}$$

and verifies

$$\int_0^1\mathcal{G}^{-1}(\tau',\tau)\mathrm{d}\tau=\frac{\mathrm{e}^{-\frac{\mathrm{i}}{2}egB\sigma_2}}{\cosh\left(\frac{\mathrm{i}egB\sigma_2}{2}\right)}\mathrm{e}^{\mathrm{i}egB\sigma_2\tau'}. \tag{26}$$

The result of the integration over ω^n is thus

$$\mathcal{I}(\theta)=\cos\left(\frac{egB}{2}\right)\exp\left[\frac{1}{2}\int_0^1\widetilde{\mathcal{J}}.\mathcal{R}^{-1}.\widetilde{\mathcal{J}}\mathrm{d}\tau\right]. \tag{27}$$

The Green function, after somewhat long calculations, is

$$\widetilde{S}_\lambda=\left(\frac{-\mathrm{i}}{2}\right)^{1-\lambda}\exp(\mathrm{i}\widetilde{\gamma}.\frac{\partial_l}{\partial\theta})\int_0^\infty\mathrm{d}e\int\mathrm{d}\chi\chi^{1-\lambda}\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\cos\left(\frac{egB}{2}\right)\sum_{n=0}^{+\infty}\left(\frac{|g|B}{\pi}\right)^{1/2}$$
$$\cdot\frac{1}{2^n n!}\mathrm{e}^{-\mathrm{i}e|g|B\left(n+\frac{1}{2}\right)}\mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)}$$
$$\cdot H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right)H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)\exp\Big[\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)$$
$$\left.+\ \mathrm{i}p_3(z_b-z_a)+\frac{\mathrm{i}e}{2}(p_0^2-p_3^2-m^2)\Big]\exp\left(\theta_\mu Q^{\mu\nu}\theta_\nu+\frac{\lambda}{2}\Upsilon_n\theta^n\chi\right)\right|_{\theta=0}, \tag{28}$$

where

$$Q^{\mu\nu} = \frac{F^{\mu\nu}}{2B}\tan\left(\frac{egB}{2}\right),\ \Upsilon_5 = m,\ \Upsilon_\mu = p_0 W_0 + p_3 W_3 + gB\frac{\left(y_b + \frac{p_1}{gB}\right)}{\sin(egB)}\left[W_2 + W_1\tan\left(\frac{egB}{2}\right)\right] - gB\frac{\left(y_a + \frac{p_1}{gB}\right)}{\sin(egB)}\left[W_2 - W_1\tan\left(\frac{egB}{2}\right)\right], \tag{29}$$

and

$$W_0 = (1,0,0,0), W_1 = (0,1,0,0), W_2 = (0,0,1,0), W_3 = (0,0,0,1)$$

are the new notations that we have adopted for convenience.

After the integration over χ, knowing that $\int \mathrm{d}\chi = 0$ and $\int \mathrm{d}\chi\chi = 1$, the Green function takes the form

$$\begin{aligned}\begin{pmatrix}\widetilde{S}_\mathrm{l}\\ G_\mathrm{g}\end{pmatrix} &= \frac{-\mathrm{i}}{2}\int_0^\infty \mathrm{d}e\cos\left(\frac{egB}{2}\right)\int\frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\begin{pmatrix}\widetilde{\Phi}_\mathrm{l}\\ \widetilde{\Phi}_\mathrm{g}\end{pmatrix}\sum_{n=0}^{+\infty}\frac{\left(\frac{|g|B}{\pi}\right)^{1/2}}{2^n n!}\mathrm{e}^{-\mathrm{i}e|g|B\left(n+\frac{1}{2}\right)}\\ &\cdot \mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)}H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right)H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)\\ &\cdot\exp\left[\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_3(z_b-z_a)+\frac{\mathrm{i}e}{2}(p_0^2-p_3^2-m^2)\right],\end{aligned} \tag{30}$$

where the spin factors are obtained with a derivation in relation to the Grassmann variables

$$\begin{pmatrix}\widetilde{\Phi}_\mathrm{l}\\ \widetilde{\Phi}_\mathrm{g}\end{pmatrix} = \exp\left(\mathrm{i}\widetilde{\gamma}^n\frac{\partial_\mathrm{l}}{\partial\theta^n}\right)\begin{pmatrix}\mathrm{i}\Upsilon_n\theta^n\\ 1\end{pmatrix}\exp(\theta_\mu Q^{\mu\nu}\theta_\nu)\Big|_{\theta=0}. \tag{31}$$

Thanks to the use of the identity

$$\exp\left(\mathrm{i}\widetilde{\gamma}^n\frac{\partial_\mathrm{l}}{\partial\theta^n}\right)f(\theta)|_{\theta=0} = f\left(\frac{\partial}{\partial\xi^n}\right)\exp(\mathrm{i}\xi_n\widetilde{\gamma}^n)\Big|_{\xi=0}, \tag{32}$$

which allows us to reduce the calculations following the method, which has been elaborated [3 – 5], the calculation result is, respectively, as follows:

$$\begin{aligned}\widetilde{\Phi}_\mathrm{l} &= -\gamma^5\left[m + \Upsilon^\mu\gamma_\mu + \mathrm{i}(\Upsilon^\alpha\gamma_\alpha)(Q^{\mu\nu}\sigma_{\mu\nu}) + 2\Upsilon_\mu Q^{\mu\nu}\gamma_\nu + \mathrm{i}mQ^{\mu\nu}\sigma_{\mu\nu} + mQ_{\mu\nu}Q^{*\mu\nu}\gamma^5\right],\\ \widetilde{\Phi}_\mathrm{g} &= 1 + \mathrm{i}Q^{\mu\nu}\sigma_{\mu\nu} + Q_{\mu\nu}Q^{*\mu\nu}\gamma^5,\end{aligned} \tag{33}$$

where $Q^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\delta}Q_{\rho\delta}$ ($\epsilon^{\mu\nu\rho\delta}$ is the Lévi-Civita tensor).

At this step it is not possible to integrate over the proper time e in the local approach case. We can put in evidence the following properties related to the harmonic oscillator $K^{\mathrm{os}}(y_b, y_a)$, which facilitate the integration:

$$\begin{aligned}\left(y_a + \frac{p_1}{gB}\right)K^{\mathrm{os}}(y_b,y_a) &= \left[\left(y_b + \frac{p_1}{gB}\right)\cos(egB) - \frac{\sin(egB)}{\mathrm{i}gB}\frac{\partial}{\partial y_b}\right]K^{\mathrm{os}}(y_b,y_a),\\ \left(y_b + \frac{p_1}{gB}\right)K^{\mathrm{os}}(y_b,y_a) &= \left[\left(y_a + \frac{p_1}{gB}\right)\cos(egB) - \frac{\sin(egB)}{\mathrm{i}gB}\frac{\partial}{\partial y_a}\right]K^{\mathrm{os}}(y_b,y_a).\end{aligned} \tag{34}$$

After development, the Polyakov factors are finally as follows:

$$\begin{aligned}\widetilde{\Phi}_\mathrm{l} &= -\gamma^5\left[m + p_0\gamma^0 + p_3\gamma^3 + gB\gamma^1\left(y_a + \frac{p_1}{gB}\right) + \mathrm{i}\gamma^2\frac{\partial}{\partial y_a}\right]\left[1 + \gamma^1\gamma^2\tan\left(\frac{egB}{2}\right)\right],\\ \widetilde{\Phi}_\mathrm{g} &= 1 + \gamma^1\gamma^2\tan\left(\frac{egB}{2}\right).\end{aligned} \tag{35}$$

Let us introduce the spin operator $S = \mathrm{i}\gamma^1\gamma^2$ and insert the identity $\sum_{s=\pm 1} \chi_s \chi_s^\dagger = 1$, where $S\chi_s = s\chi_s$, to extract the energy spectrum for the two representations. Let us as well use the relation $S^{\mathrm{c}} = -\widetilde{S}\gamma^5$, which gives the Green function relative to the Dirac equation without γ^5.

The Green function relative to our particle in the local and global representations is finally given by the following expressions:

$$\begin{aligned}\begin{pmatrix} \mathbf{S}_1^{\mathrm{c}} \\ G_{\mathrm{g}} \end{pmatrix} &= \frac{-\mathrm{i}}{2} \int \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi} \begin{pmatrix} \mathcal{F} \\ 1 \end{pmatrix} \int_0^\infty \mathrm{d}e \sum_{s=\pm 1}\sum_{n=0}^{+\infty} \left\{ \left(\frac{|g|B}{\pi}\right)^{1/2} \frac{1}{2^n n} \mathrm{e}^{-\mathrm{i}e|g|B\left(n+\frac{1}{2}\right)} \right. \\ &\quad \cdot \mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)} \exp\Big[\mathrm{i}p_0(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_3(z_b-z_a) \\ &\quad \left. +\frac{\mathrm{i}e}{2}(p_0^2-p_3^2-m^2-gsB)\Big] H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right) H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)\chi_s\chi_s^\dagger \right\},\end{aligned} \tag{36}$$

where

$$\mathcal{F} = p_0\gamma^0 + gB\gamma^1\left(y_a+\frac{p_1}{gB}\right) + \mathrm{i}\gamma^2\frac{\partial}{\partial y_a} + p_3\gamma^3 - m. \tag{37}$$

Integrating over e:

$$\begin{aligned}\begin{pmatrix} \mathbf{S}_1^{\mathrm{c}} \\ G_{\mathrm{g}} \end{pmatrix} &= \int \frac{\mathrm{d}p_0}{2\pi}\frac{\mathrm{d}p_1}{2\pi}\frac{p_3}{2\pi} \begin{pmatrix} \mathcal{F} \\ 1 \end{pmatrix}\left(\frac{|g|B}{\pi}\right)^{1/2} \sum_{s=\pm 1}\sum_{n=0}^{+\infty}\left\{\frac{1}{2^n n!}\mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)} \exp\big[\mathrm{i}p_0(t_b-t_a)\right. \\ &\quad \left. +\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_3(z_b-z_a)\big] \frac{H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right)H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)}{p_0^2-\mathcal{E}^2}\chi_s\chi_s^\dagger\right\},\end{aligned} \tag{38}$$

we obtain the same energy spectrum as follows:

$$\mathcal{E} = \sqrt{p_3^2 + m^2 + gsB + 2gB\left(n+\frac{1}{2}\right)} \tag{39}$$

for the two cases: local and global.

To finally extract the corresponding wave functions, let us integrate over p_0. We thus obtain the following spectral decomposition:

$$\begin{aligned}\begin{pmatrix} \mathbf{S}_1^{\mathrm{c}} \\ G_{\mathrm{g}} \end{pmatrix} &= \int \frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\begin{pmatrix} \mathcal{F} \\ 1 \end{pmatrix}\sum_{s=\pm 1}\sum_{n=0}^{+\infty}\left\{\frac{\left(\frac{|g|B}{\pi}\right)^{1/2}}{2^n n!}\mathrm{e}^{-\frac{|g|B}{2}\left(\left(y_b+\frac{p_1}{gB}\right)^2+\left(y_a+\frac{p_1}{gB}\right)^2\right)}\right. \\ &\quad \left.\cdot\exp\left[\mathrm{i}\mathcal{E}(t_b-t_a)+\mathrm{i}p_1(x_b-x_a)+\mathrm{i}p_3(z_b-z_a)\right]\frac{H_n\left(\sqrt{|g|B}\left(y_b+\frac{p_1}{gB}\right)\right)H_n\left(\sqrt{|g|B}\left(y_a+\frac{p_1}{gB}\right)\right)}{2\mathcal{E}}\chi_s\chi_s^\dagger\right\} \\ &= \sum_{s=-1}^{s=+1}\sum_{n=0}^{+\infty}\int\int\frac{\mathrm{d}p_1}{2\pi}\frac{\mathrm{d}p_3}{2\pi}\begin{pmatrix}\Psi^{\mathrm{l}}_{p_1,p_3,n,s}(x_b,y_b,z_b;t_b)\Psi^{\mathrm{l}\dagger}_{p_1,p_3,n,s}(x_a,y_a,z_a;t_a)\gamma^0 \\ \Psi^{\mathrm{g}}_{p_1,p_3,n,s}(x_b,y_b,z_b;t_b)\Psi^{\mathrm{g}\dagger}_{p_1,p_3,n,s}(x_a,y_a,z_a;t_a)\end{pmatrix},\end{aligned} \tag{40}$$

and the wave functions in the local and global cases corresponding to the energies are, respectively,

$$\begin{aligned}\Psi^{\mathrm{l}}_{p_1,p_3,n,s}(x,y,z;t) &= \frac{1}{\sqrt{2\mathcal{E}}}\left[-\mathcal{E}\gamma^0-\gamma^1(p_1+gBy)+\mathrm{i}\gamma^2\frac{\partial}{\partial y}-\gamma^3p_3+m\right] \\ &\quad\cdot\frac{\left(\frac{|g|B}{\pi}\right)^{1/4}}{\sqrt{2^n n!}}\mathrm{e}^{-\frac{|g|B}{2}\left(y+\frac{p_1}{gB}\right)^2}H_n\left(\sqrt{|g|B}\left(y+\frac{p_1}{gB}\right)\right)\mathrm{e}^{\mathrm{i}\mathcal{E}t+\mathrm{i}p_1x+\mathrm{i}p_3z}\chi_s,\end{aligned} \tag{41}$$

$$\Psi^{g}_{p_1,p_3,n,s}(x,y,z;t) = \frac{1}{\sqrt{2\mathcal{E}}}\frac{\left(\frac{|g|B}{\pi}\right)^{1/4}}{\sqrt{2^n n!}}\mathrm{e}^{-\frac{|g|B}{2}\left(y+\frac{p_1}{gB}\right)^2} H_n\left(\sqrt{|g|B}\left(y+\frac{p_1}{gB}\right)\right) \cdot \exp\left[\mathrm{i}\mathcal{E}t + \mathrm{i}p_1 x + \mathrm{i}p_3 z\right]\chi_s. \tag{42}$$

We note finally that $\Psi^{\mathrm{l}}_{p_1,p_3,n,s}(x,y,z;t)$ can be obtained starting from $\Psi^{\mathrm{g}}_{p_1,p_3,n,s}(x,y,z;t)$:

$$\Psi^{\mathrm{l}}_{p_1,p_3,n,s}(x,y,z;t) = (\gamma^{\mu}(\mathrm{i}\partial_{\mu} - gA_{\mu}) + m)\Psi^{\mathrm{g}}_{p_1,p_3,n,s}(x,y,z;t). \tag{43}$$

The result agrees with the literature [8].

3. Conclusion

We have obtained the energy spectrum of a Dirac particle in motion in a constant magnetic field through an analytical and an exact manner [8]. We have obtained as well the corresponding wave functions following two different manners, local and global, by using the supersymmetric representation of Fradkin and Gitman for the Green function. We should note, however, that the spin factor in the local approach has never been extracted before.

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